

Well-posedness of the extrusion model described by coupled hyperbolic systems with a free boundary

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Abstract

In this paper, we consider the well-posedness of the Cauchy problem for a physical model of the extrusion process, which is described by two systems of conservation laws with a free boundary. By suitable change of coordinates and fixed point argument, we obtain the existence, uniqueness and regularity of the weak solution to this Cauchy problem.

Keywords: Conservation law, free boundary, well-posedness, extruder model.

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1 Introduction

Balance equations provide the foundation for much physical-based modeling in fluid dynamics. They are also the starting point for developing qualitative understanding of phenomenological observations in fluid mechanics, heat transfer, mass transfer, and reaction engineering.

The mathematical analysis of mobile interfaces in the context of moving boundary problems has been an active subject in the last decades and their mathematical understanding continues to be an important interdisciplinary tool for the scientific applications. Such kind of partial differential equation (PDE) model arises in many applications devoted to modeling of biological systems and reaction diffusion processes which involve stefan problems, crystal growth processes. One can mention applications which concern swelling nanocapsules [4], lyophilization [33], cooking processes [30], freeze drying process modeling [13], mixing systems (model of torus reactor including a well-mixed zone and a transport zone), diesel oxidation catalyst [29].

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Concerning biological systems, [7] proposes an analysis of the global existence of solutions to a coupled parabolic-hyperbolic system with moving boundary representing cell mobility. A similar type of nonlinear moving-boundary problem, consisting of a hyperbolic equation and a parabolic equation for modeling blood flow through viscoelastic arteries [6] and tumor growth [12], is studied in terms of well-posedness. Mathematical study of coupled partial differential equations through an internal moving interface is also proposed in [28] for parabolic systems and in [11, 31] for hyperbolic systems, where the PDEs are defined in a time-varying spatial domain.

Generally speaking, the key resolution for such problems is based on a suitable change of coordinates which transforms the system with moving interface into a system defined on a fixed domain. Then, the results of many studies dedicated to systems of conservation laws may be useful to establish the existence, uniqueness, regularity and continuous dependence of solutions. For the well-posedness problems, we refer to the works [1, 5, 22, 27] (and the references therein) in the content of weak solutions to systems (including scalar case) of conservation laws, and [25, 26] in the content of classical solutions to general quasi-linear hyperbolic systems. Recall that there exists a classical approach developed for fixed interfaces, consisting in augmenting the hyperbolic system of conservation laws with color functions [16, 17] for numerical analysis. For both cases with fixed or moving interfaces, the lack of physical models which express clearly the interface structure and the coupling conditions can be considered as the real challenge from modeling point of view. An interfacial model which describes precisely the information that are exchanged at the coupling region should be defined with respect to the real physical constraints.

In this paper we consider the well-posedness of the Cauchy problem for a physical model of the extrusion process. The process model is composed by heat and mass transport equations which are defined in complementary time-varying spatial domains. The domains are coupled by a moving interface whose dynamics is governed by an ordinary differential equation (ODE) expressing the conservation of mass in an extruder. More detailed description of the model is given in Section 2. We mention that the first result concerning the mathematical analysis of the extrusion model as transport equations coupled via complementary time varying domains is proposed in [15], where the well-posedness for the linearized model of the extruder is obtained by using perturbation theory on the linear operator.

Our proof of the well-posedness of the Cauchy problem for the extrusion model relies on a change of coordinates and a fixed point argument (see **Theorem 4.1**). To tackle with the difficulty caused by the moving interface, we make suitable change of coordinates on the spatial variables so that the moving interface problem is normalized to a standard fixed domain problem. In order to deal with the nonlinearities, we use Banach fixed point theorem based on the characteristic method, which enables to compute the solution numerically. The H^2 -regularity of the solution is proved as well (see **Theorem 4.2**), which is useful when one considers the asymptotic stabilization of the corresponding closed-loop system with feedback

controls (see, e.g., [9] for hyperbolic systems with boundary feedback laws).

We point out here that the analysis of the Cauchy problem for the extrusion model is fundamental to numerical simulation of the physical process. Moreover, it is also the first step for further study on control of the model. It is of particular interest to consider some control problems, including controllability and stabilization of the filling ratio, the net flow rate, the position of the interface, the moisture and the temperature of the extrusion process. These problems will be studied in some forthcoming papers.

The organization of this paper is as follows: First in Section 2, we give a description of the extrusion process model which is derived from conservation laws. As preliminaries, we make domain normalization by change of coordinates on space variables in Section 3. The main results (**Theorems 4.1, 4.2, 4.3**) concerning the well-posedness and regularity of the normalized system are presented in Section 4, while their proofs are given in Section 5-6 respectively. Finally, in Section 7, we give the conclusion of this paper as well as some perspectives.

2 Description of the extrusion process model

Extruders are designed to process highly viscous materials. They are mainly used in the chemical industries for polymer processing as well as in the food industries. An extruder is made of a barrel, the temperature of which is regulated. One or two Archimedean screws are rotating inside the barrel. The extruder is equipped with a die where the material comes out of the process (see Fig 1). The extruder is of particular interest due to its modular geometry that allows the control of capacities of the mixtures along the machine. Another interesting property of the extruder is that the filling ratio along the axial direction of the screws can be less than one in some part of the system according to the screw configuration and the operating conditions. For modeling purposes, the main phenomenon is obviously the fluid flow which may be considered as highly viscous Newtonian or non-Newtonian fluid flows interacting with heat transfer and possibly chemical reactions. These processes occur within a complex non-stationary volume delimited by the barrel and the rotating screw. The most important part of the extruder is the screw configuration which modulates extensively the mechanical energy.

The complexity of screw geometric configuration in an extruder make difficult the design a non-isothermal flow model [3]. In [2], an analysis of the flow in the channel of co-rotating twin-screws at the same speed is developed and the authors show how a reasonable flow analysis can be made by writing a single screw extrusion process as an equivalent model. In an extruder, the net flow at the die exit is mainly due to the flow of the material in the longitudinal direction if one neglects the clearance between the screw and the barrel and the vibrations which can occur due to screws structure. Therefore, the flow dynamics which is

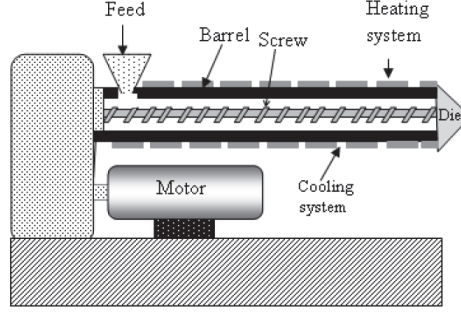


Fig. 1: Schematic description of an extruder

dominated by the convection effect in the direction of screw axis is sufficient to represent the material flow. This means that the transverse flow corresponding to a recirculation of the material in the plane perpendicular to the screw channel is neglected. From a macroscopic point of view, a 1D model describes clearly the material convection aspect in an extruder. So, the material is driven from the feed to the die by the pumping effect of the screw rotation. The geometric structure of the die influences the transport along the extruder. Therefore, the material is accumulated behind the die and fills completely the available volume at this region. The spatial domain where the extruder is completely filled is called the *Fully Filled Zone (FFZ)*. The flow in this *FFZ* depends on the pumping capacity of the screw and also on the pressure flow. The pressure gradient which appears due to the die restriction is given by Navier-Stokes equation which provides a mathematical model of the fluid motion. The extruder which is initially empty, may also comprises a spatial domain that is not completely filled by the material. This region which corresponds to a conveying region is called *Partially Filled Zone (PFZ)*. In this domain, there is no pressure build-up. This means that the pressure gradient is zero and the pressure is generally equal to the air pressure inside the barrel. The transport velocity of the material is controlled by the screw speed. These two zones are coupled by an interface which characterizes the spatial domains where the pressure gradient is null or not (and accordingly where the extruder is partially or fully filled). The mobile interface is assumed to be thin, i.e., reduced to a point. In the sequel, the spatial domain of the extruder will be taken to be the real interval $[0, L]$ where $L > 0$ is the length of the extruder. Let us denote by $l(t) \in [0, L]$ the position of the thin interface, the domain of the *PFZ* is then $[0, l(t)]$ and the *FFZ* is defined on $[l(t), L]$, see Fig 2. The interface is moving according to the volume of the material which is accumulated in the *FFZ*. It is clear that the interface which separates these two zones evolves as a function of the difference between the feed and die rates. Finally, the extruder model is composed by three interdependent dynamics which describe the evolution of the material in the *PFZ* and *FFZ* and the evolution of the position of the interface.

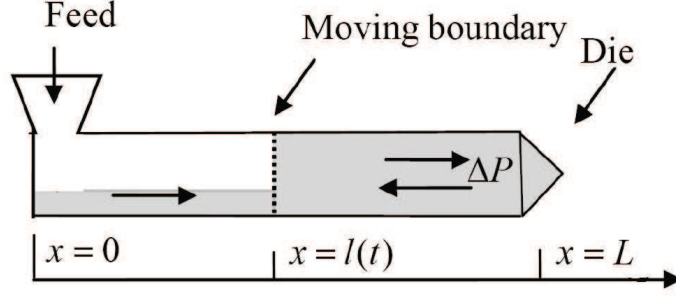


Fig. 2: Bi-zone model of an extruder

The extrusion model is based on this structural decomposition of the extruder and derived from the mass and the energy balances as in [21, 23]. The transport equations describe the evolution of the filling ratio, the moisture contains and the temperature for an extruded material. The problem of coupled PDEs through an moving interface arises from the existence of transport equations which are defined in two complementary time-varying spatial domains denoted by $[0, l(t)]$ and $[l(t), L]$.

2.1 Physical definition of the parameters

L	Extruder Length	B	Geometric parameter
F_d	Net forward mass flow rate	K_d	Geometric parameter
S_{ech}	Exchange area between melt and barrel	V_{eff}	Effective volume
α	Heat exchange coefficient	S_{eff}	Effective area
c_o	Specific heat capacity	η	Melt viscosity
β_o	Coefficient of viscous heat generation	ρ_o	Melt density
μ_p, μ_f	Viscous heat generation factor	ζ	Screw Pitch

2.2 The Partially Filled Zone (PFZ) $[0, T] \times [0, l(t)]$

For the *PFZ*, we consider the filling ratio f_p , the moisture content M_p and the temperature T_p as the state variables. The transport equations associated with these variables are defined on $[0, T] \times [0, l(t)]$, where $l(t)$ represents the moving interface between the two zones. The filling ratio f_p and the moisture M_p are strictly positive functions which are less than one according to the modeling assumptions. The balance equations express the convection through the rotation of the screw at translational velocity α_p , product on the pitch of the screw ζ and the rotation speed of the screw $N(t)$. The source term Ω_p which appears in the equation of T_p groups the heat produced by the rotational screw (proportional to $N^2(t)$) and

the heat exchange with the barrel ($T_b(t, x)$ is the distributed barrel temperature).

$$\partial_t \begin{pmatrix} f_p \\ M_p \\ T_p \end{pmatrix} = -\alpha_p \partial_x \begin{pmatrix} f_p \\ M_p \\ T_p \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Omega_p \end{pmatrix}, \quad (2.1)$$

where

$$\alpha_p = \zeta N(t) \quad (2.2)$$

$$\Omega_p = \frac{\mu_p \beta_o \eta N^2(t)}{f_p(t, x) \rho_o V_{eff} c_o} + \frac{\zeta S_{ech} \alpha}{\rho_o V_{eff} c_o} (T_b(t, x) - T_p(t, x)). \quad (2.3)$$

2.3 The Fully Filled Zone (FFZ) $[0, T] \times [l(t), L]$

For the *FFZ*, we consider the moisture content M_f and the temperature T_f as variables of the states. The transport equations associated with these variables are defined on $[0, T] \times [l(t), L]$, where L is the length of the extruder. The transport velocity α_f is proportional to the net flow rate $F_d(t)$ at the die. Similarly as in the *PFZ*, the source term Ω_f stands for the heat produced by the rotational screw and the heat exchange with the barrel in the *FFZ*.

$$\partial_t \begin{pmatrix} M_f \\ T_f \end{pmatrix} = -\alpha_f \partial_x \begin{pmatrix} M_f \\ T_f \end{pmatrix} + \begin{pmatrix} 0 \\ \Omega_f \end{pmatrix}, \quad (2.4)$$

where

$$\alpha_f = \frac{\zeta F_d(t)}{\rho_o V_{eff}} \quad (2.5)$$

$$\Omega_f = \frac{\mu_f \beta_o \eta N^2(t)}{\rho_o V_{eff} c_o} + \frac{\zeta S_{ech} \alpha}{\rho_o V_{eff} c_o} (T_b(t, x) - T_f(t, x)). \quad (2.6)$$

2.4 The interface

Following [19, 20, 21, 23], we assume that the two zones are separated by an interface defined by the discontinuity of the filling ratio. By assuming the continuity of pressure at the interface $l(t)$, we express the net flow rate F_d as a function of $l(t)$ and $N(t)$ as following:

$$F_d(t) = \frac{K_d}{\eta} \Delta P(t), \quad (2.7)$$

$$\Delta P(t) := P(t, L) - P_0 = \frac{\eta \rho_o V_{eff} N(t) (L - l(t))}{B \rho_o + K_d (L - l(t))}. \quad (2.8)$$

The interface position is physically determined by the variation of pressure from the *PFZ* to the *FFZ* and thus its dynamic is generated by the gradient of pressure which appears in the *FFZ*. The equation (2.8) is actually obtained by integrating the pressure gradient relation from $l(t)$ to L derived from momentum balance in the *FFZ* (where B is a coefficient of pressure flow)

$$\partial_x P(t, x) = \eta \frac{\rho_o V_{eff} N(t) - F_d(t)}{B \rho_o}. \quad (2.9)$$

We emphasize that without assuming a constant viscosity η , which is not distributed with respect to the space variable, along the extruder, the analytical solution of (2.9) can not be computed. Numerical simulations has been performed by [23] considering the viscosity η as a distributed function of moisture and temperature for steady state profile. In this case, the mass flow and the moisture are constant and the moving interface is stationary. The author uses iterative schemes assuming that the pressure and the temperature profiles are known at the first step of the computation. The simulation shows the dynamics of the shaft power which is the power consumed by viscous dissipation in the *PFZ* and the *FFZ* and the power required to force the material through the die.

The interface dynamics which arises from a total mass balance is given by the following ODE

$$\begin{cases} \dot{l}(t) = F(l(t), N(t), f_p(t, l(t))), \\ l(0) = l^0, \end{cases} \quad (2.10)$$

where

$$F(l(t), N(t), f_p(t, l(t))) = \frac{F_d(t) - \rho_o V_{eff} N(t) f_p(t, l(t))}{\rho_o S_{eff} (1 - f_p(t, l(t)))}. \quad (2.11)$$

Recall that in the *PFZ*, the filling ratio satisfies $f_p(t, x) < 1$, $x \in (0, l(t))$ with $f_p(t, l(t)) < 1$ and in the *FFZ*, $f_f(t, x) = 1$, $x \in (l(t), L)$.

The moisture and the temperature are assumed to be continuous at the interface

$$(M_p(t, l(t)), T_p(t, l(t)))^{tr} = (M_f(t, l(t)), T_f(t, l(t)))^{tr}. \quad (2.12)$$

2.5 Initial and boundary conditions

The initial conditions are given as the following

$$(f_p(0, x), M_p(0, x), T_p(0, x))^{tr} = (f_p^0(x), M_p^0(x), T_p^0(x))^{tr}, \quad x \in (0, l^0), \quad (2.13)$$

$$(M_f(0, x), T_f(0, x))^{tr} = (M_f^0(x), T_f^0(x))^{tr}, \quad x \in (l^0, L). \quad (2.14)$$

The boundary conditions are given as the following

$$(f_p(t, 0), M_p(t, 0), T_p(t, 0))^{tr} = \left(\frac{F_{in}(t)}{\theta(N(t))}, M_{in}(t), T_{in}(t) \right)^{tr}, \quad (2.15)$$

where $F_{in}(t)$ denotes the feed rate and

$$\theta(N(t)) = \rho_o V_{eff} N(t). \quad (2.16)$$

Since the flow occurs in the direction of the screw channel, θ is the maximum pumping capacity of the screw.

3 Domain normalization of the extrusion process model

In this section, we will transform, by change of coordinates on space variables, the original system with free boundary (Cauchy problem (2.1), (2.4), (2.10), (2.12), (2.13), (2.14) and (2.15)) to a normalized system with fixed boundary.

For the *PFZ* zone, after change of variable

$$y = \frac{x}{l(t)}$$

from $(0, l(t))$ onto the interval $(0, 1)$ (see [14]), we normalize system (2.1) to a new system defined on $Q := (0, T) \times (0, 1)$. For the sake of simplicity, we still denote by x the space variable instead of y , the unknown functions by (f_p, M_p, T_p) , the velocity by α_p and the source term by Ω_p . We have for all $(t, x) \in Q$ that

$$\partial_t \begin{pmatrix} f_p(t, x) \\ M_p(t, x) \\ T_p(t, x) \end{pmatrix} + \alpha_p(x, N(t), l(t), f_p(t, 1)) \partial_x \begin{pmatrix} f_p(t, x) \\ M_p(t, x) \\ T_p(t, x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \Omega_p(t, x, T_p(t, x)) \end{pmatrix}, \quad (3.1)$$

with

$$\alpha_p(x, N(t), l(t), f_p(t, 1)) = \frac{1}{l(t)} (\zeta N(t) - x F(l(t), N(t), f_p(t, 1))), \quad (3.2)$$

$$\Omega_p(t, x, T_p(t, x)) = \textcolor{red}{C}_o(T_p(t, x) - T_b(t, x)) + g_p(t, x), \quad (3.3)$$

where

$$\textcolor{red}{C}_o = -\frac{\zeta S_{ech} \alpha}{\rho_o V_{eff} \textcolor{red}{c}_o}, \quad g_p(t, x) = \frac{\mu_p \beta_o \eta N^2(t)}{f_p(t, x) \rho_o V_{eff} \textcolor{red}{c}_o}. \quad (3.4)$$

For the *FFZ* zone, after change of variable

$$y = \frac{x - l(t)}{L - l(t)}$$

from $(l(t), L)$ onto the interval $(0, 1)$ (see [14]). system (2.4) can be normalized to a new system defined on Q . For the sake of simplicity, we still denote by x the space variable instead of y , the unknown functions by (M_f, T_f) , the velocity term by α_f and the source term by Ω_f . Then we have for all $(t, x) \in Q$ that

$$\partial_t \begin{pmatrix} M_f(t, x) \\ T_f(t, x) \end{pmatrix} + \alpha_f(x, N(t), l(t), f_p(t, 1)) \partial_x \begin{pmatrix} M_f(t, x) \\ T_f(t, x) \end{pmatrix} = \begin{pmatrix} 0 \\ \Omega_f(t, x, T_f(t, x)) \end{pmatrix}, \quad (3.5)$$

with

$$\alpha_f(x, N(t), l(t), f_p(t, 1)) = \frac{1}{L - l(t)} \left(\frac{\zeta F_d(t)}{\rho_o V_{eff}} + (x - 1) F(l(t), N(t), f_p(t, 1)) \right), \quad (3.6)$$

$$\Omega_f(t, x, T_f(t, x)) = \textcolor{red}{C}_o(T_f(t, x) - T_b(t, x)) + g_f(t), \quad (3.7)$$

where

$$F_d(t) = \frac{K_d \rho_o V_{eff} N(t) (L - l(t))}{B \rho_o + K_d (L - l(t))}, \quad (3.8)$$

$$C_o = -\frac{\zeta S_{ech} \alpha}{\rho_o V_{eff} c_o}, \quad g_f(t) = \frac{\mu_f \beta_o \eta N^2(t)}{\rho_o V_{eff} c_o}. \quad (3.9)$$

The boundary conditions (2.12) and (2.15) can be rewritten as

$$(M_f(t, 0), T_f(t, 0))^{tr} = (M_p(t, 1), T_p(t, 1))^{tr}, \quad (3.10)$$

$$(f_p(t, 0), M_p(t, 0), T_p(t, 0))^{tr} = \left(\frac{F_{in}(t)}{\rho_o V_{eff} N(t)}, M_{in}(t), T_{in}(t) \right)^{tr}. \quad (3.11)$$

In summary, we consider a coupled system composed of an ODE for the moving interface

$$\begin{cases} \dot{l}(t) = F(l(t), N(t), f_p(t, 1)), & t \in (0, T), \\ l(0) = l^0, \end{cases} \quad (3.12)$$

a transport equation for the filling ratio in the *PFZ*

$$\begin{cases} \partial_t f_p(t, x) + \alpha_p(x, N(t), l(t), f_p(t, 1)) \partial_x f_p(t, x) = 0, & (t, x) \in Q, \\ f_p(0, x) = f_p^0(x), & x \in (0, 1), \\ f_p(t, 0) = \frac{F_{in}(t)}{\rho_o V_{eff} N(t)}, & t \in (0, T), \end{cases} \quad (3.13)$$

two transport equations for the moisture

$$\begin{cases} \partial_t M_p(t, x) + \alpha_p(x, N(t), l(t), f_p(t, 1)) \partial_x M_p(t, x) = 0, & (t, x) \in Q, \\ \partial_t M_f(t, x) + \alpha_f(x, N(t), l(t), f_p(t, 1)) \partial_x M_f(t, x) = 0, & (t, x) \in Q, \\ M_p(0, x) = M_p^0(x), \quad M_f(0, x) = M_f^0(x), & x \in (0, 1), \\ M_p(t, 0) = M_{in}(t), \quad M_f(t, 0) = M_p(t, 1), & t \in (0, T), \end{cases} \quad (3.14)$$

and two transport equations for the temperature

$$\begin{cases} \partial_t T_p(t, x) + \alpha_p(x, N(t), l(t), f_p(t, 1)) \partial_x T_p(t, x) = \Omega_p(t, x, T_p(t, x)), & (t, x) \in Q, \\ \partial_t T_f(t, x) + \alpha_f(x, N(t), l(t), f_p(t, 1)) \partial_x T_f(t, x) = \Omega_f(t, x, T_f(t, x)), & (t, x) \in Q, \\ T_p(0, x) = T_p^0(x), \quad T_f(0, x) = T_f^0(x), & x \in (0, 1), \\ T_p(t, 0) = T_{in}(t), \quad T_f(t, 0) = T_p(t, 1), & t \in (0, T). \end{cases} \quad (3.15)$$

Remark 3.1. *Nonlinearity of the model is focused on the Cauchy problem (3.12)-(3.13) which is closed for (l, f_p) . With known values of (l, f_p) , Cauchy problem (3.14)-(3.15) is linear with respect to the unknowns (M_p, M_f, T_p, T_f) .*

In the whole paper, unless otherwise specified, we always assume that $l^0 \in (0, L)$, $f_p^0 \in W^{1,\infty}(0, 1)$, $M_p^0, T_p^0, M_f^0, T_f^0 \in L^2(0, 1)$, $M_{in}, T_{in} \in L^2(0, T)$, $F_{in}, N \in W^{1,\infty}(0, T)$ and $T_b \in L^2(Q)$. For the sake of simplicity, we denote from now on $\|f\|_{L^\infty}$ ($\|f\|_{W^{1,\infty}}$, $\|f\|_{L^2}$, resp.) as the L^∞ ($W^{1,\infty}$, L^2 , resp.) norm of the function f with respect to its variables.

4 Main Results

In this section, we show the main results on the well-posedness of the whole coupled system (3.12)-(3.15). We first study the nonlinear Cauchy problem (3.12)-(3.13) since it is closed for (l, f_p) , then turn to linear Cauchy problem (3.14)-(3.15) with known (l, f_p) .

Concerning the Cauchy problem (3.12)-(3.13), we have the following two theorems

Theorem 4.1. *Let $T > 0$. Let (l_e, N_e, f_{pe}) be a constant equilibrium, i.e.,*

$$F(l_e, N_e, f_{pe}) = 0 \quad (4.1)$$

with $0 < f_{pe} < 1$, $0 < l_e < L$. Assume that the compatibility condition at $(0, 0)$ holds

$$\frac{F_{in}(0)}{\rho_o V_{eff} N(0)} = f_p^0(0). \quad (4.2)$$

Then, there exists ε_0 (depending on T) such that for any $\varepsilon \in (0, \varepsilon_0]$, if

$$\|f_p^0(\cdot) - f_{pe}\|_{W^{1,\infty}} + \left\| \frac{F_{in}(\cdot)}{\rho_o V_{eff} N(\cdot)} - f_{pe} \right\|_{W^{1,\infty}} + \|N(\cdot) - N_e\|_{W^{1,\infty}} + |l^0 - l_e| \leq \varepsilon, \quad (4.3)$$

Cauchy problem (3.12)-(3.13) admits a unique solution $(l, f_p) \in W^{1,\infty}(0, T) \times W^{1,\infty}(Q)$, and the following estimates hold

$$\|f_p(\cdot, \cdot) - f_{pe}\|_{W^{1,\infty}} \leq C_{\varepsilon_0} \cdot \varepsilon, \quad (4.4)$$

$$\|l(\cdot) - l_e\|_{W^{1,\infty}} \leq C_{\varepsilon_0} \cdot \varepsilon, \quad (4.5)$$

where C_{ε_0} is a constant depending on ε_0 , but independent of ε .

Theorem 4.2. *Under the assumptions of Theorem 4.1, we assume furthermore that $f_p^0(\cdot) \in H^2(0, 1)$, $\frac{F_{in}(\cdot)}{\rho_o V_{eff} N(\cdot)} \in H^2(0, T)$, and the compatibility condition at $(0, 0)$ holds*

$$(f_p^0)_x(0) + \frac{l(0)}{\zeta N(0)} \cdot \frac{F'_{in}(0)N(0) - F_{in}(0)N'(0)}{\rho_o V_{eff} N^2(0)} = 0. \quad (4.6)$$

Then, there exists ε_0 (depending on T) such that for any $\varepsilon \in (0, \varepsilon_0]$, if

$$\|f_p^0(\cdot) - f_{pe}\|_{H^2(0,1)} + \left\| \frac{F_{in}(\cdot)}{\rho_o V_{eff} N(\cdot)} - f_{pe} \right\|_{H^2(0,T)} + \|N(\cdot) - N_e\|_{W^{1,\infty}} + |l^0 - l_e| \leq \varepsilon, \quad (4.7)$$

Cauchy problem (3.12)-(3.13) has a unique solution $(l, f_p) \in W^{1,\infty}(0, T) \times C^0([0, T]; H^2(0, 1))$ with the additional estimate

$$\|f_p(\cdot, \cdot) - f_{pe}\|_{C^0([0,T]; H^2(0,1))} \leq C_{\varepsilon_0} \cdot \varepsilon, \quad (4.8)$$

where C_{ε_0} is a constant depending on ε_0 , but independent of ε .

Remark 4.1. *The solution in Theorem 4.1 or in Theorem 4.2 is often called semi-global solution since it exists on any preassigned time interval $[0, T]$ if (l, f_p) has some kind of smallness (depending on T), see [24, 34].*

Remark 4.2. We have the hidden regularity that $f_p \in C^0([0, 1]; H^2(0, T))$ in Theorem 4.2.

For the proof of Remark 4.2, one can refer to [10, 32].

Concerning the moisture equation (3.14) and the temperature equation (3.15), we have the following theorem

Theorem 4.3. Under the assumptions of Theorem 4.1, Cauchy problem (3.14)-(3.15) admits a unique solution $(M_p, M_f, T_p, T_f) \in (C^0([0, T]; L^2(0, 1)))^4$, and the following estimates hold

$$\|M_p\|_{C^0([0, T]; L^2(0, 1))} \leq C \cdot (\|M_p^0\|_{L^2} + \|M_{in}\|_{L^2}), \quad (4.9)$$

$$\|T_p\|_{C^0([0, T]; L^2(0, 1))} \leq C \cdot (\|T_p^0\|_{L^2} + \|T_{in}\|_{L^2} + \|g_p\|_{L^2}), \quad (4.10)$$

$$\|M_f\|_{C^0([0, T]; L^2(0, 1))} \leq C \cdot (\|M_p^0\|_{L^2} + \|M_{in}\|_{L^2} + \|M_f^0\|_{L^2}), \quad (4.11)$$

$$\|T_f\|_{C^0([0, T]; L^2(0, 1))} \leq C \cdot (\|T_p^0\|_{L^2} + \|T_{in}\|_{L^2} + \|T_f^0\|_{L^2} + \|g_f\|_{L^2}), \quad (4.12)$$

where g_p and g_f are defined as (3.4) and (3.9) respectively and $C > 0$ is a constant.

5 Proof of Theorem 4.1

In order to conclude Theorem 4.1, it suffices to prove the following lemma on local well-posedness of Cauchy problem (3.12)-(3.13).

Lemma 5.1. There exist $\varepsilon_1 > 0$ suitably small and $\delta = \delta(\varepsilon_1, \|f_p^0(\cdot) - f_{pe}\|_{W^{1, \infty}}, |l^0 - l_e|) > 0$, such that for any $\varepsilon \in (0, \varepsilon_1]$, $f_p^0 \in W^{1, \infty}(0, 1)$, $F_{in}, N \in W^{1, \infty}(0, T)$, $l^0 \in (0, L)$ with

$$\|f_p^0(\cdot) - f_{pe}\|_{W^{1, \infty}} + \left\| \frac{F_{in}(\cdot)}{\rho_o V_{eff} N(\cdot)} - f_{pe} \right\|_{W^{1, \infty}} + \|N(\cdot) - N_e\|_{W^{1, \infty}} + |l^0 - l_e| \leq \varepsilon, \quad (5.1)$$

Cauchy problem (3.12)-(3.13) admits a unique local solution on $[0, \delta]$, which satisfies the following estimates

$$\|f_p(t, \cdot) - f_{pe}\|_{W^{1, \infty}} \leq C_{\varepsilon_1} \cdot \varepsilon, \quad \forall t \in [0, \delta], \quad (5.2)$$

$$|l(t) - l_e| \leq C_{\varepsilon_1} \cdot \varepsilon, \quad \forall t \in [0, \delta], \quad (5.3)$$

where C_{ε_1} is a constant depending on ε_1 , but independent of ε .

Let us first show how to conclude Theorem 4.1 from Lemma 5.1. By Lemma 5.1, we take $\varepsilon_2 \in (0, \varepsilon_1]$ such that $C_{\varepsilon_1} \cdot \varepsilon_2 \leq \varepsilon_1$. Then for any $\varepsilon \in (0, \varepsilon_2]$ and any initial-boundary data such that (5.1) holds, Cauchy problem (3.12)-(3.13) admits a unique solution on $[0, \delta]$. Furthermore, one has

$$\|f_p(\delta, \cdot) - f_{pe}\|_{W^{1, \infty}} \leq C_{\varepsilon_1} \cdot \varepsilon \leq \varepsilon_1, \quad (5.4)$$

$$|l(\delta) - l_e| \leq C_{\varepsilon_1} \cdot \varepsilon \leq \varepsilon_1. \quad (5.5)$$

By taking $(l(\delta), f_p(\delta, \cdot))$ as new initial data and applying Lemma 5.1 on $[\delta, 2\delta]$, the solution of Cauchy problem (3.12)-(3.13) is extended to $[0, 2\delta]$. For fixed $T > 0$, we can extend the

local solution to Cauchy problem (3.12)-(3.13) to $[0, T]$ eventually by reducing the value of ε_0 and applying Lemma 5.1 in finite times (at most $[T/\delta] + 1$ times). Therefore, to conclude Theorem 4.1, it remains to prove Lemma 5.1. \square

Proof of Lemma 5.1: The proof is divided into 4 steps.

Step 1. Existence and uniqueness of $(l(\cdot), f_p(\cdot, 1))$ by fixed point argument.

Let $\varepsilon_1 > 0$ be such that

$$0 < \varepsilon_1 < \min\{l_e, L - l_e, f_{pe}, 1 - f_{pe}\}. \quad (5.6)$$

Denote

$$\|F\|_{W^{1,\infty}} := \sum_{|\alpha| \leq 1} \operatorname{ess\,sup}_{\substack{0 < x_1 < L \\ N_e - \varepsilon_1 < x_2 < N_e + \varepsilon_1 \\ 0 < x_3 < 1}} |D^\alpha F(x_1, x_2, x_3)|, \quad (5.7)$$

$$\Psi(t) := (l(t), f_p(t, 1)), \quad t \in [0, T]. \quad (5.8)$$

For any given $\delta > 0$ small enough (to be chosen later), we define a domain candidate as a closed subset of $C^0([0, \delta])$ with respect to C^0 norm:

$$\Omega_{\delta, \varepsilon_1} := \left\{ \Psi \in C^0([0, \delta]) : \Psi(0) = (l^0, f_p^0(1)), \|\Psi(\cdot) - (l_e, f_{pe})\|_{C^0([0, \delta])} \leq \varepsilon_1 \right\}. \quad (5.9)$$

We denote by $\xi(s; t, x)$, with $(s, \xi(s; t, x)) \in [0, t] \times [0, 1]$ the characteristic curve passing through the point $(t, x) \in [0, \delta] \times [0, 1]$ (see Fig 3), i.e.,

$$\begin{cases} \frac{d\xi(s; t, x)}{ds} = \alpha_p(\xi(s; t, x), N(s), l(s), f_p(s, 1)), \\ \xi(t; t, x) = x. \end{cases} \quad (5.10)$$

Let us define a map $\mathfrak{F} := (\mathfrak{F}_1, \mathfrak{F}_2)$, where $\mathfrak{F} : \Omega_{\delta, \varepsilon_1} \rightarrow C^0([0, \delta])$, $\Psi \mapsto \mathfrak{F}(\Psi)$ as

$$\mathfrak{F}_1(\Psi)(t) := l^0 + \int_0^t F(l(s), N(s), f_p(s, 1)) ds, \quad (5.11)$$

$$\mathfrak{F}_2(\Psi)(t) := f_p^0(\xi(0; t, 1)). \quad (5.12)$$

Solving the linear ODE (5.10) with α_p given by (3.2), one easily gets for all δ small and all $0 \leq s \leq t \leq \delta$ that

$$\xi(s; t, 1) = e^{\int_s^t \frac{F(l(\sigma), N(\sigma), f_p(\sigma, 1))}{l(\sigma)} d\sigma} - \int_s^t \frac{\zeta N(\sigma)}{l(\sigma)} \cdot e^{\int_s^\sigma \frac{F(l(s), N(s), f_p(s, 1))}{l(s)} ds} d\sigma. \quad (5.13)$$

It is obvious that \mathfrak{F} maps into $\Omega_{\delta, \varepsilon_1}$ itself if

$$0 < \delta < \min \left\{ T_0, T, \frac{l_e - \varepsilon_1}{\|F\|_{W^{1,\infty}}}, \frac{L - l_e - \varepsilon_1}{\|F\|_{W^{1,\infty}}} \right\}, \quad (5.14)$$

where T_0 denotes the time when the characteristic curve $\xi(s; 0, 0)$ arrives at $x = 1$, i.e., $\xi(T_0; 0, 0) = 1$.

Now we prove that, if δ is small enough, \mathfrak{F} is a contraction mapping on $\Omega_{\delta, \varepsilon_1}$ with respect to the C^0 norm. Let $\Psi = (l, f_p)$, $\bar{\Psi} = (\bar{l}, \bar{f}_p) \in \Omega_{\delta, \varepsilon_1}$. We denote by $\bar{\xi}(s; t, x)$ the corresponding characteristic curve passing through (t, x) :

$$\begin{cases} \frac{d\bar{\xi}(s; t, x)}{ds} = \alpha_p(\bar{\xi}(s; t, x), N(s), \bar{l}(s), \bar{f}_p(s, 1)), \\ \bar{\xi}(t; t, x) = x. \end{cases} \quad (5.15)$$

Similarly as (5.13), one has for all δ small and all $0 \leq s \leq t \leq \delta$ that

$$\bar{\xi}(s; t, 1) = e^{\int_s^t \frac{F(\bar{l}(\sigma), N(\sigma), \bar{f}_p(\sigma, 1))}{\bar{l}(\sigma)} d\sigma} - \int_s^t \frac{\zeta N(\sigma)}{\bar{l}(\sigma)} \cdot e^{\int_s^\sigma \frac{F(\bar{l}(s), N(s), \bar{f}_p(s, 1))}{\bar{l}(s)} ds} d\sigma. \quad (5.16)$$

Therefore it holds for all $t \in [0, \delta]$ that

$$\begin{aligned} |\mathfrak{F}_1(\bar{\Psi})(t) - \mathfrak{F}_1(\Psi)(t)| &= \left| \int_0^t F(\bar{l}(s), N(s), \bar{f}_p(s, 1)) ds - \int_0^t F(l(s), N(s), f_p(s, 1)) ds \right| \\ &\leq \delta \|F\|_{W^{1, \infty}} \|\bar{\Psi} - \Psi\|_{C^0([0, \delta])}. \end{aligned} \quad (5.17)$$

On the other hand, it follows from (5.12), (5.13) and (5.16), that for all $t \in [0, \delta]$,

$$\begin{aligned} &|\mathfrak{F}_2(\bar{\Psi})(t) - \mathfrak{F}_2(\Psi)(t)| \\ &= |f_p^0(\bar{\xi}(0; t, 1)) - f_p^0(\xi(0; t, 1))| \\ &\leq \|f_{px}^0\|_{L^\infty} \left\{ \left| e^{\int_0^t \frac{F(\bar{l}(\sigma), N(\sigma), \bar{f}_p(\sigma, 1))}{\bar{l}(\sigma)} d\sigma} - e^{\int_0^t \frac{F(l(\sigma), N(\sigma), f_p(\sigma, 1))}{l(\sigma)} d\sigma} \right| \right. \\ &\quad \left. + \int_0^t \left| \frac{\zeta N(\sigma)}{\bar{l}(\sigma)} \cdot e^{\int_0^\sigma \frac{F(\bar{l}(s), N(s), \bar{f}_p(s, 1))}{\bar{l}(s)} ds} - \frac{\zeta N(\sigma)}{l(\sigma)} \cdot e^{\int_0^\sigma \frac{F(l(s), N(s), f_p(s, 1))}{l(s)} ds} \right| d\sigma \right\}. \end{aligned}$$

By (5.7) and the fact that $\Psi, \bar{\Psi} \in \Omega_{\delta, \varepsilon_1}$ and $l(t) \geq l_e - \varepsilon_1 > 0, |N(t)| \leq N_e + \varepsilon_1$, it follows that for all $t \in [0, \delta]$,

$$\begin{aligned} &|\mathfrak{F}_2(\bar{\Psi})(t) - \mathfrak{F}_2(\Psi)(t)| \\ &\leq \|f_{px}^0\|_{L^\infty} \left\{ e^{\frac{\delta \|F\|_{W^{1, \infty}}}{l_e - \varepsilon_1}} \int_0^t \left| \frac{F(\bar{l}(\sigma), N(\sigma), \bar{f}_p(\sigma, 1))}{\bar{l}(\sigma)} - \frac{F(l(\sigma), N(\sigma), f_p(\sigma, 1))}{l(\sigma)} \right| d\sigma \right. \\ &\quad + \frac{\zeta(N_e + \varepsilon_1)}{l_e - \varepsilon_1} e^{\frac{\delta \|F\|_{W^{1, \infty}}}{l_e - \varepsilon_1}} \int_0^t \int_0^\sigma \left| \frac{F(\bar{l}(s), N(s), \bar{f}_p(s, 1))}{\bar{l}(s)} - \frac{F(l(s), N(s), f_p(s, 1))}{l(s)} \right| ds d\sigma \\ &\quad \left. + \zeta(N_e + \varepsilon_1) e^{\frac{\delta \|F\|_{W^{1, \infty}}}{l_e - \varepsilon_1}} \int_0^t \left| \frac{1}{\bar{l}(\sigma)} - \frac{1}{l(\sigma)} \right| d\sigma \right\} \\ &\leq \|f_{px}^0\|_{L^\infty} e^{\frac{\delta \|F\|_{W^{1, \infty}}}{l_e - \varepsilon_1}} \left\{ \delta \|F\|_{W^{1, \infty}} \left(1 + \frac{\delta \zeta(N_e + \varepsilon_1)}{l_e - \varepsilon_1} \right) \left[\frac{\|\bar{\Psi} - \Psi\|_{C^0([0, \delta])}}{l_e - \varepsilon_1} + \frac{\|\bar{l} - l\|_{C^0([0, \delta])}}{(l_e - \varepsilon_1)^2} \right] \right. \\ &\quad \left. + \delta \zeta(N_e + \varepsilon_1) \frac{\|\bar{l} - l\|_{C^0([0, \delta])}}{(l_e - \varepsilon_1)^2} \right\} \\ &\leq \delta e^{\frac{T \|F\|_{W^{1, \infty}}}{l_e - \varepsilon_1}} \left\{ \left(1 + \frac{T \zeta(N_e + \varepsilon_1)}{l_e - \varepsilon_1} \right) \left[\frac{\|F\|_{W^{1, \infty}}}{l_e - \varepsilon_1} + \frac{\|F\|_{W^{1, \infty}}}{(l_e - \varepsilon_1)^2} \right] + \frac{\zeta(N_e + \varepsilon_1)}{(l_e - \varepsilon_1)^2} \right\} \\ &\quad \cdot \|f_p^0(\cdot) - f_{pe}\|_{W^{1, \infty}} \|\bar{\Psi} - \Psi\|_{C^0([0, \delta])}. \end{aligned} \quad (5.18)$$

Finally, we choose δ small enough (depending on ε_1 , $\|f_p^0(\cdot) - f_{pe}\|_{W^{1,\infty}}$, $\|F\|_{W^{1,\infty}}$) such that

$$\begin{aligned} & \|\mathfrak{F}(\bar{\Psi}) - \mathfrak{F}(\Psi)\|_{C^0([0,\delta])} \\ &:= \max \left\{ \|\mathfrak{F}_1(\bar{\Psi}) - \mathfrak{F}_1(\Psi)\|_{C^0([0,\delta])}, \|\mathfrak{F}_2(\bar{\Psi}) - \mathfrak{F}_2(\Psi)\|_{C^0([0,\delta])} \right\} \\ &\leq \frac{1}{2} \|\bar{\Psi} - \Psi\|_{C^0([0,\delta])}, \end{aligned} \quad (5.19)$$

Banach fixed point theorem implies the existence of the unique fixed point $(l(\cdot), f_p(\cdot, 1))$ of the mapping \mathfrak{F} : $\Psi = \mathfrak{F}(\Psi)$ in $\Omega_{\delta, \varepsilon_1}$.

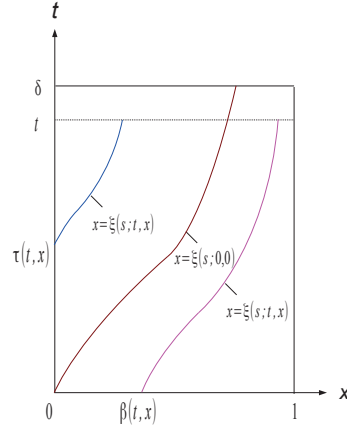


Fig. 3: The characteristics $\xi(s; t, x)$

Step 2. Construction of a solution by characteristic method.

With the existence of $(l(\cdot), f_p(\cdot, 1))$ and $\delta > 0$ by **Step 1**, we can construct a solution to Cauchy problem (3.12)-(3.13). For every (t, x) in $[0, \delta] \times [0, 1]$, we still denote by $\xi(s; t, x)$, with $(s, \xi(s; t, x)) \in [0, t] \times [0, 1]$, the characteristic curve passing through the point (t, x) ; see (5.10) and Fig.3. Since the velocity function α_p is positive, the characteristic $\xi(s; t, x)$ intersects the x -axis at point $(0, \beta(t, x))$ with $\beta(t, x) = \xi(0; t, x)$ if $0 \leq \xi(t; 0, 0) \leq x \leq 1$; the characteristic $\xi(s; t, x)$ intersects the t -axis at point $(\tau(t, x), 0)$ with $\xi(\tau(t, x); t, x) = 0$ if $0 \leq x \leq \xi(t; 0, 0)$. Moreover, we have (see [26, Lemma 3.2 and its proof, Page 90-91] for a more general situation)

$$\frac{\partial \tau(t, x)}{\partial t} = \frac{-l(\tau(t, x))}{\zeta N(\tau(t, x))} \frac{\zeta N(t) - xF(l(t), N(t), f_p(t, 1))}{l(t)} e^{\int_{\tau(t, x)}^t \frac{F(l(\sigma), N(\sigma), f_p(\sigma, 1))}{l(\sigma)} d\sigma}, \quad (5.20)$$

$$\frac{\partial \tau(t, x)}{\partial x} = \frac{-l(\tau(t, x))}{\zeta N(\tau(t, x))} e^{\int_{\tau(t, x)}^t \frac{F(l(\sigma), N(\sigma), f_p(\sigma, 1))}{l(\sigma)} d\sigma}, \quad (5.21)$$

$$\frac{\partial \beta(t, x)}{\partial t} = -\frac{\zeta N(t) - xF(l(t), N(t), f_p(t, 1))}{l(t)} e^{\int_0^t \frac{F(l(\sigma), N(\sigma), f_p(\sigma, 1))}{l(\sigma)} d\sigma}, \quad (5.22)$$

$$\frac{\partial \beta(t, x)}{\partial x} = e^{\int_0^t \frac{F(l(\sigma), N(\sigma), f_p(\sigma, 1))}{l(\sigma)} d\sigma}. \quad (5.23)$$

We define f_p by

$$f_p(t, x) = \begin{cases} \frac{F_{in}(\tau(t, x))}{\rho_o V_{eff} N(\tau(t, x))}, & \text{if } 0 \leq x \leq \xi(t; 0, 0) \leq 1, 0 \leq t \leq \delta, \\ f_p^0(\beta(t, x)), & \text{if } 0 \leq \xi(t; 0, 0) \leq x \leq 1, 0 \leq t \leq \delta. \end{cases} \quad (5.24)$$

Then it is easy to check that $(l, f_p) \in W^{1,\infty}(0, \delta) \times W^{1,\infty}((0, \delta) \times (0, 1))$ under the compatibility condition (4.2) and (l, f_p) is indeed a solution to Cauchy problem (3.12)-(3.13).

Step 3. Uniqueness of the solution.

Assume that Cauchy problem (3.12)-(3.13) has two solutions $(l, f_p), (\bar{l}, \bar{f}_p)$ on $[0, \delta] \times [0, 1]$. It follows that $(l(\cdot), f_p(\cdot, 1)) = (\bar{l}(\cdot), \bar{f}_p(\cdot, 1))$ since they are both the fixed point of the mapping $\mathfrak{F}: \Psi = \mathfrak{F}(\Psi)$ in $\Omega_{\delta, \varepsilon_1}$. This fact implies that the characteristics $\xi(\cdot; t, x)$ and $\bar{\xi}(\cdot; t, x)$ coincide with each other and therefore so do the solutions f_p and \bar{f}_p by characteristic method.

Step 4. A priori estimate on the local solution.

By definition of f_p and assumption (5.1), it is clear that for all $t \in [0, \delta]$,

$$\|f_p(t, \cdot) - f_{pe}\|_{L^\infty} \leq \varepsilon. \quad (5.25)$$

Thanks to (3.12), (4.1), (5.25) and assumption (5.1), we get for all $t \in [0, \delta]$ that

$$\begin{aligned} |\dot{l}(t)| &= |F(l(t), N(t), f_p(t, 1)) - F(l_e, N_e, f_{pe})| \\ &\leq \|F\|_{W^{1,\infty}}(|l(t) - l_e| + |N(t) - N_e| + |f_p(t, 1) - f_{pe}|). \end{aligned} \quad (5.26)$$

$$\leq \|F\|_{W^{1,\infty}}|l(t) - l_e| + 2\varepsilon\|F\|_{W^{1,\infty}}, \quad (5.27)$$

which yields (5.3) from (5.1) and Gronwall's inequality. On the other hand,

$$\begin{aligned} \left\| \frac{\partial f_p}{\partial x} \right\|_{L^\infty} &\leq \left\| \frac{\partial}{\partial x} \left(\frac{F_{in}(\tau(t, x))}{\rho_o V_{eff} N(\tau(t, x))} \right) \right\|_{L^\infty} + \left\| \frac{\partial}{\partial x} f_p^0(\beta(t, x)) \right\|_{L^\infty} \\ &\leq \left\| \frac{F_{in}(\cdot)}{\rho_o V_{eff} N(\cdot)} - f_{pe} \right\|_{W^{1,\infty}} \left\| \frac{\partial \tau}{\partial x} \right\|_{L^\infty} + \left\| f_p^0(\cdot) - f_{pe} \right\|_{W^{1,\infty}} \left\| \frac{\partial \beta}{\partial x} \right\|_{L^\infty}. \end{aligned} \quad (5.28)$$

Combining (5.21), (5.23), (5.28) and assumption (5.1), we obtain (5.2) which concludes the proof of Lemma 5.1. \square

6 Proof of Theorem 4.2 and Theorem 4.3

Before proving Theorem 4.2 and Theorem 4.3, let us recall a classical result on Cauchy problem of the following general linear transport equation

$$\begin{cases} u_t + a(t, x)u_x = b(t, x)u + c(t, x), & (t, x) \in Q = (0, T) \times (0, 1), \\ u(0, x) = u_0(x), & x \in (0, 1), \\ u(t, 0) = h(t), & t \in (0, T), \end{cases} \quad (6.1)$$

where $a(t, x) > 0$, $a, a_x, b \in L^\infty(Q)$ and $c \in L^2(Q)$.

We recall from [8, Section 2.1], the definition of a weak solution to Cauchy problem (6.1).

Definition 6.1. Let $T > 0$, $u_0 \in L^2(0, 1)$, $h \in L^2(0, T)$ be given. A weak solution of Cauchy problem (6.1) is a function $u \in C^0([0, T]; L^2(0, 1))$ such that for every $\tau \in [0, T]$, every test function $\varphi \in C^1([0, T] \times [0, 1])$ such that $\varphi(t, 1) = 0$, $\forall t \in [0, T]$, one has

$$\begin{aligned} & - \int_0^\tau \int_0^1 \left(u[\partial_t \varphi + a \partial_x \varphi + (a_x + b)\varphi] + c\varphi \right) dx dt + \int_0^1 u(\tau, \cdot) \varphi(\tau, \cdot) dx \\ & - \int_0^1 u_0 \varphi(0, \cdot) dx - \int_0^\tau h a(\cdot, 0) \varphi(\cdot, 0) dt = 0. \end{aligned} \quad (6.2)$$

We have the following lemma

Lemma 6.1. Let $T > 0$, $u_0 \in L^2(0, 1)$ and $h \in L^2(0, T)$ be given. Then, Cauchy problem (6.1) has a unique weak solution $u \in C^0([0, T]; L^2(0, 1))$ and the following estimate holds:

$$\|u\|_{C^0([0, T]; L^2(0, 1))} \leq C(\|u_0\|_{L^2(0, 1)} + \|h\|_{L^2(0, T)} + \|c\|_{L^2(Q)}), \quad (6.3)$$

where $C = C(T, \|a\|_{L^\infty(Q)}, \|a_x\|_{L^\infty(Q)}, \|b\|_{L^\infty(Q)})$ is a constant independent of u_0, h, c .

For the proof of Lemma 6.1, one can refer to [26] for classical solution or [18, Theorem 23.1.2, Page 387] for Cauchy problem on \mathbb{R} without boundary.

Proof of Theorem 4.2. By Theorem 4.1 and Lemma 6.1, it suffices to prove that the systems of $f_{p_{xx}}$ satisfies all the assumptions of Lemma 6.1.

Differentiating (3.13) with respect to x once and twice give us successively that

$$\begin{cases} \partial_t f_{p_x}(t, x) + \alpha_p(t, x) \partial_x f_{p_x}(t, x) = -\alpha_{p_x}(t, x) f_{p_x}(t, x), & (t, x) \in Q, \\ f_{p_x}(0, x) = f_{p_x}^0(x), & x \in (0, 1), \\ f_{p_x}(t, 0) = \frac{-l(t)}{\zeta N(t)} \cdot \left(\frac{F_{in}(t)}{\rho_o V_{eff} N(t)} \right)', & t \in (0, T), \end{cases} \quad (6.4)$$

and

$$\begin{cases} \partial_t f_{p_{xx}}(t, x) + \alpha_p(t, x) \partial_x f_{p_{xx}}(t, x) = -2\alpha_{p_x}(t, x) f_{p_{xx}}(t, x), & (t, x) \in Q, \\ f_{p_{xx}}(0, x) = f_{p_{xx}}^0(x), & x \in (0, 1), \\ f_{p_{xx}}(t, 0) = \frac{-l(t)}{\zeta N(t)} \left[\frac{F(l(t), N(t), f_p(t, 1))}{\zeta N(t)} \left(\frac{F_{in}(t)}{\rho_o V_{eff} N(t)} \right)' \right. \\ \quad \left. - \left(\frac{l(t)}{\zeta N(t)} \left(\frac{F_{in}(t)}{\rho_o V_{eff} N(t)} \right)' \right)' \right], & t \in (0, T), \end{cases} \quad (6.5)$$

with

$$\alpha_p(t, x) = \frac{\zeta N(t) - x F(l(t), N(t), f_p(t, 1))}{l(t)}, \quad \alpha_{p_x}(t, x) = \frac{-F(l(t), N(t), f_p(t, 1))}{l(t)}. \quad (6.6)$$

From the assumptions that $f_p^0 \in H^2(0, 1)$, $\frac{F_{in}(\cdot)}{\rho_o V_{eff} N(\cdot)} \in H^2(0, T)$ and the compatibility conditions (4.2) and (4.6), one easily concludes Theorem 4.2 by applying Lemma 6.1 to Cauchy problem (6.5). \square

Proof of Theorem 4.3. By Theorem 4.1, we have already $(l, f_p) \in W^{1, \infty}(0, T) \times W^{1, \infty}(Q)$. Then Theorem 4.3 is a direct consequence of Lemma 6.1 by solving first (M_p, T_p) in the PFZ and next (M_f, T_f) in the FFZ . \square

7 Conclusion

In this paper, we consider the well-posedness of the Cauchy problem for a physical model of the extrusion process, which is described by two systems of conservation laws in complementary time varying domains. After a suitable change of coordinates in space variables the original system is transformed into a normalized problem in fixed domain. Then we prove the existence and uniqueness of $(l(t), f_p(t, 1))$ for t small by Banach fixed point theorem. With the known $(l(t), f_p(t, 1))$, we construct a local solution to this Cauchy problem and prove that the local solution is unique. Using the estimates on the local solution and induction in time, we can extend the local solution to the semi-global one. The solution we obtained in Theorem 4.1 is called semi-global solution since it exists on any preassigned time interval $[0, T]$ if the initial and boundary data has some kind of smallness (depending on T). The H^2 -regularity of the filling ratio f_p is also proved as preliminaries for asymptotic stabilization for the corresponding closed-loop system with feedback controls.

Based on the analytical results obtained in this paper, we are able to study the controllability and stabilization of this model which is important in applications. It is interesting, in particular, to study the controllability of boundary profile, i.e., to reach the desired moisture and temperature at the die under suitable controls. These interesting control problems will be studied in some forthcoming papers. We point out also that the assumption of constant fluid viscosity along the extruder allows to decouple the interface dynamics to moisture and temperature of the mixture. However, for many extruded material the fluid viscosity may significantly change with the chemical composition and temperature evolutions. Our future works also consist in analysis and control of this extrusion process model in the case of distributed viscosity. The study on these problem is really challenging for mathematical analysis but also more useful in applications.

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References

- [1] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math.*, 161(1):223–342, 2005.
- [2] M. L. Booy. Geometry of fully wiped twin-screw equipment. *Polymer Engineering & Science*, 18(12):973–984, 1978.
- [3] M. L. Booy. Isothermal flow of viscous liquids in corotating twin screw devices. *Polymer Engineering Science*, 20:1220–1228, 1980.
- [4] K. Bouchemal, F. Couenne, S. Briancon, H. Fessi, and M. Tayakout. Polyamides nanocapsules : Modeling and wall thickness estimation. *AiChE*, 52:2161–2170, 2006.
- [5] Alberto Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.
- [6] Suncica Canic, Andro Mikelic, Tae-Beom Kim, and Giovanna Guidoboni. *Existence of a Unique Solution to a Nonlinear Moving-Boundary Problem of Mixed Type Arising in Modeling Blood Flow*, volume 153. Springer US, 2011.
- [7] Y. S. Choi and Craig Miller. Global existence of solutions to a coupled parabolic-hyperbolic system with moving boundary. In *proceedings of the American Mathematical Society*, volume 139, page 3257?3270, 2011.
- [8] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.
- [9] Jean-Michel Coron, Georges Bastin, and Brigitte d’Andréa Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM J. Control Optim.*, 47(3):1460–1498, 2008.
- [10] Jean-Michel Coron, Matthias Kowski, and Zhiqiang Wang. Analysis of a conservation law modeling a highly re-entrant manufacturing system. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1337–1359, 2010.
- [11] D. Coutand and S. Shkoller. Well-posedness in smooth function spaces for moving-boundary 1-d compressible euler equations in physical vacuum. *Communications on pure and applied*, 64:328–366, 2011.

- [12] Chen D. and Friedman A. A two-phase free boundary problem with discontinuous velocity: Application to tumor model. *Journal of Mathematical Analysis and Applications*, 1(399):378–393, 2013.
- [13] N. Daraoui, P. Dufour, H. Hammouri, and A. Hottot. Model predictive control during the primary drying stage of lyophilisation. *Control Engineering Practice*, 18(5):483–494, 2010.
- [14] M. Diagne, V. Dos Santos Martins, F. Couenne, and B. Maschke. Well posedness of the model of an extruder in infinite dimension. In *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, pages 1311–1316, 2011.
- [15] M. Diagne, V. Dos Santos Martins, F. Couenne, B. Maschke, and C. Jallut. Modélisation et commande d’un système d’équations aux dérivées partielles à frontière mobile : application au procédé d’extrusion. *Journal Européen des Systèmes Automatisés*, 45:665–691, 2011.
- [16] E. Godlewski, K.-C. Le Than, and P.A. Raviart. The numerical interface coupling of nonlinear systems of conservation laws: II. the case of systems. *ESAIM: Mathematical Modelling and Numerical Analysis*, 39(4):649–692, 2005.
- [17] E. Godlewski and P.-A. Raviart. The numerical interface coupling of nonlinear hyperbolic systems of conservation laws: I. the scalar case. *Numer. Math.*, 97:81–130, 2004.
- [18] Lars Hörmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1994. Pseudo-differential operators, Corrected reprint of the 1985 original.
- [19] E. K. Kim and J. L. White. Isothermal transient startup for starved flow modular co-rotating twin screw extruder. *Polymer Engineering and Science*, 40:543–553, 2004.
- [20] E. K. Kim and J. L. White. Non-isothermal transient startup for starved flow modular co-rotating twin screw extruder. *International Polymer Processing*, 15:233–241, 2004.
- [21] M.K. Kulshrestha and C.A. Zaror. An unsteady state model for twin screw extruders. *Tran IChemE, PartC*, 70:21–28, 1992.
- [22] Philippe G. LeFloch. *Hyperbolic systems of conservation laws*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2002. The theory of classical and nonclassical shock waves.
- [23] Chin-Hsien Li. Modelling extrusion cooking. *Mathematical and Computer Modelling*, 33:553–563, 2001.

- [24] Ta-tsien Li and Yi Jin. Semi-global c_1 solution to the mixed initial-boundary value problem for quasilinear hyperbolic systems. *Chinese Ann. Math. Ser. B*, 22:325–336, 2001.
- [25] Tatsien Li. *Global classical solutions for quasilinear hyperbolic systems*. Research in Applied Mathematics **32**, John Wiley & Sons, Chichester, 1994.
- [26] Tatsien Li and Wenci Yu. *Boundary value problems for quasilinear hyperbolic systems*. Duke University Mathematics Series, V. Duke University Mathematics Department, Durham, NC, 1985.
- [27] Taiping Liu and Tong Yang. Well-posedness theory for hyperbolic conservation laws. *Comm. Pure Appl. Math.*, 52(12):1553–1586, 1999.
- [28] A. Muntean. Well-posedness of a moving-boundary problem with two moving reaction strips. *Nonlinear Analysis: Real World Applications*, 10:2541–2557, 2009.
- [29] N. Petit. Control problems for one-dimensional fluids and reactive fluids with moving interfaces. In *Advances in the theory of control, signals and systems with physical modeling*, volume 407 of *Lecture notes in control and information sciences*, pages 323–337, Lausanne, Dec 2010.
- [30] E. Purlis and V. O. Salvadori. A moving boundary problem in a food material undergoing volumechange - simulation of bread baking. *Food Research International*, 43:949–958, 2008.
- [31] R. Raul Borsche, R. M. Colombo, and M. Garavello. On the coupling of systems of hyperbolic conservation laws with ordinary differential equations. *Nonlinearity*, 23, 2010.
- [32] Peipei Shang and Zhiqiang Wang. Analysis and control of a scalar conservation law modeling a highly re-entrant manufacturing system. *J. Differential Equations*, 250(2):949–982, 2011.
- [33] S. A. Velardi and A. A. Barresi. Development of simplified models for the freeze-drying process and investigation of the optimal operating conditions. *Chemical Engineering Research and Design*, 86:9–22, 2008.
- [34] Zhiqiang Wang. Exact controllability for nonautonomous first order quasilinear hyperbolic systems. *Chinese Ann. Math. Ser. B*, 27:643–656, 2006.